LVI. *A Discourse on the* Locus *for three and four Lines celebrated among the ancient Geometers, by* H. Pemberton, *M. D. R. S.* Lond. *et R. A.* Berol. *S. In a Letter to the Reverend* Thomas Birch, *D. D. Secretary to the Royal Society.*

SIR, Dec. 15. 1763.

Read at R. S. 15 Dec. 1763.

MY worthy friend, and associate in my early studies, the collector of the late Mr. Robins's mathematical tracts, thought it conducive to a more compleat vindication of the memory of his friend against an insinuation prejudicial to his candour, to make some mention of the course, I took in my early mathematical pursuits, and how soon I became attached to the ancient manner of treating geometrical subjects. This gave occasion to my looking into some of my old papers, amongst which I found a discussion of the problem relating to the *locus ad tres & quatuor lineas* celebrated among the ancients, which I then communicated to a friend or two, whose sentiments of those ancient sages were the same with mine. What I had drawn up on this subject is contained in the papers, I herewith put into your hands, which if you shall think worthy of being laid before our honourable society, they are intirely at your disposal.

I am your most obedient servant,

H. Pemberton.

THE

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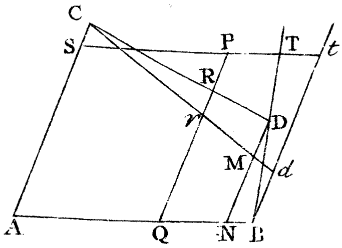

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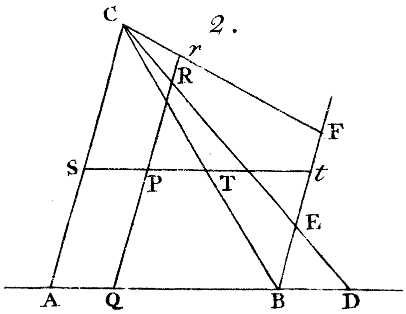

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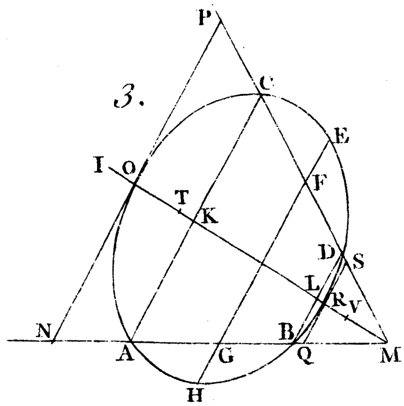

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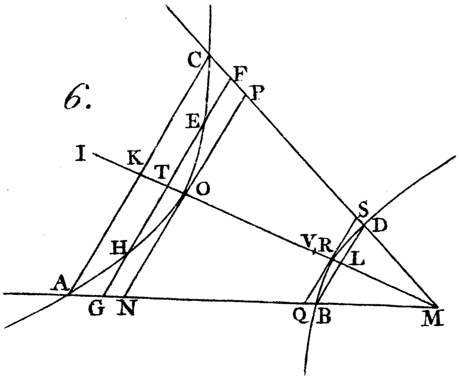

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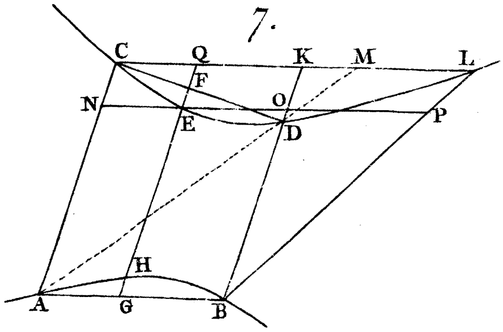

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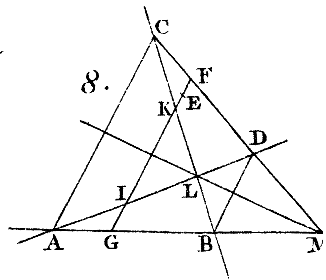

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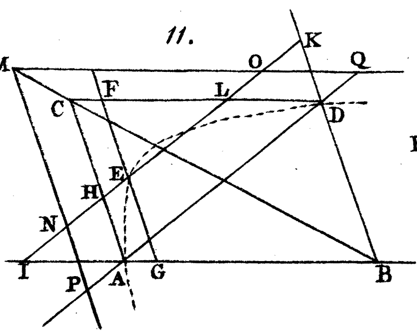

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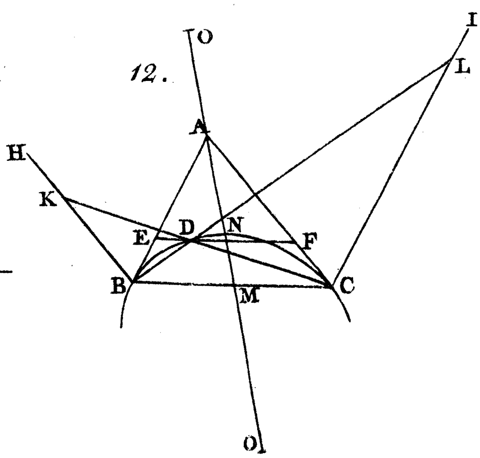

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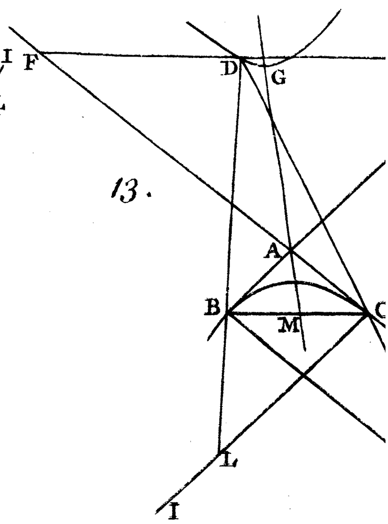

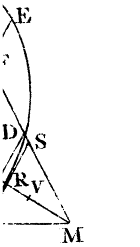
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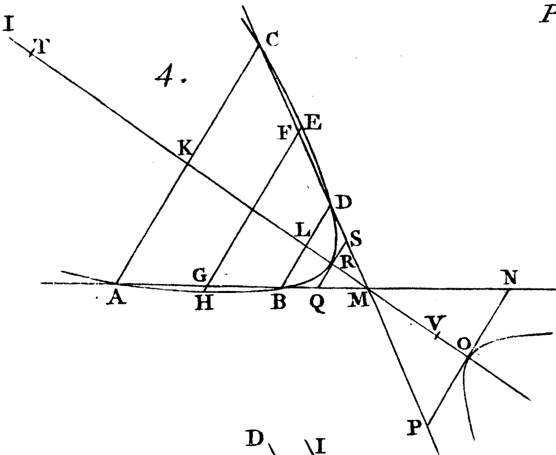
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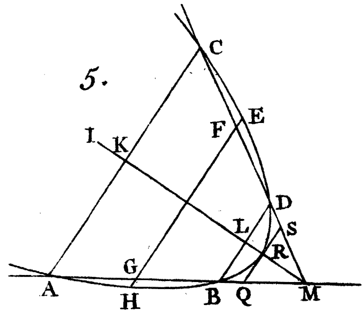
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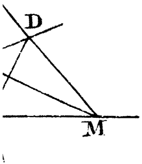
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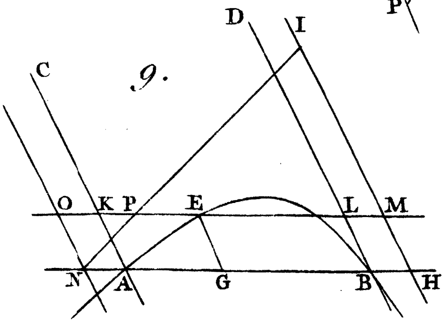
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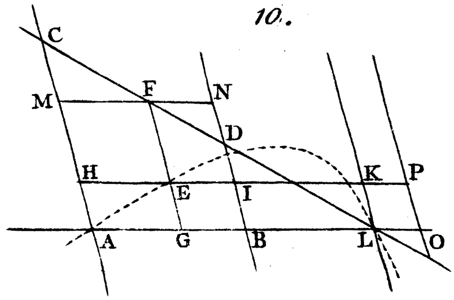
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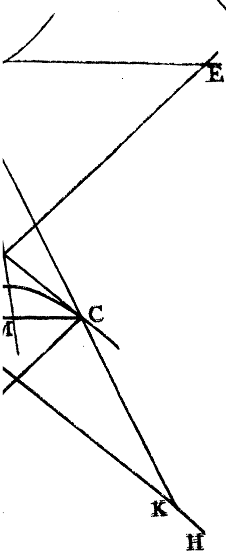
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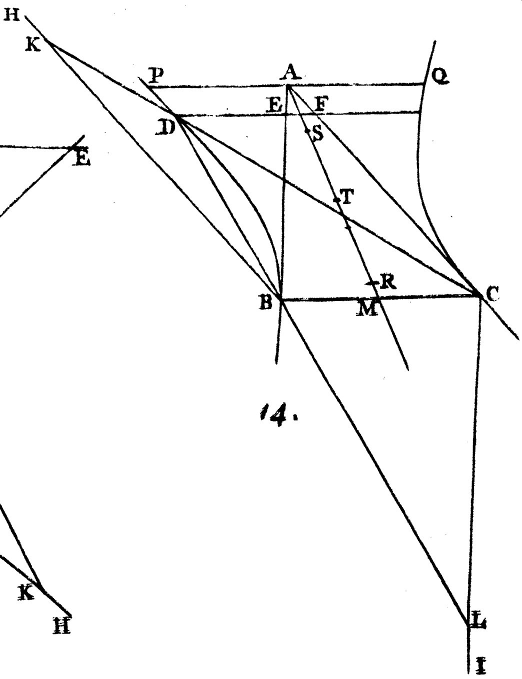
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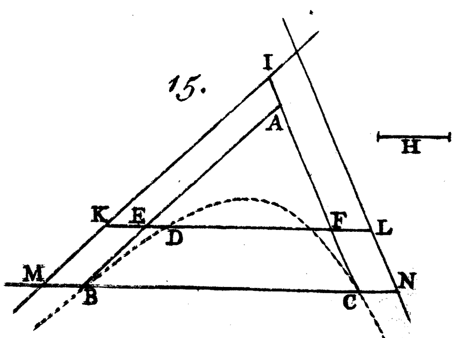
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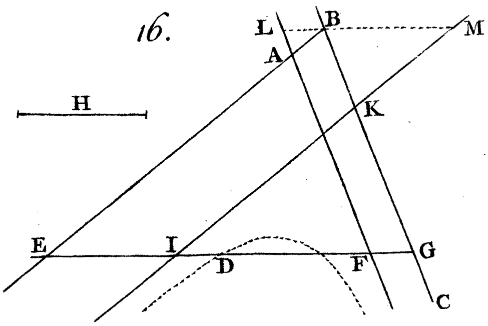
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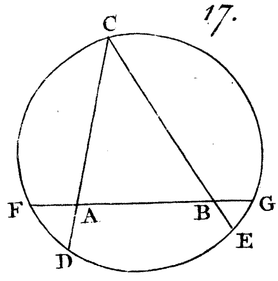
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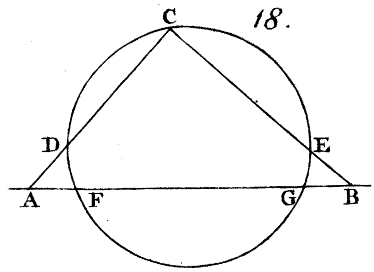
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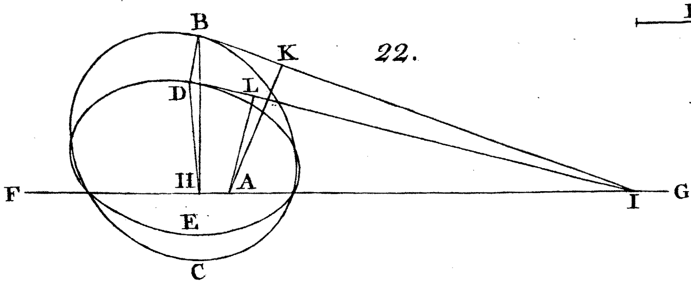
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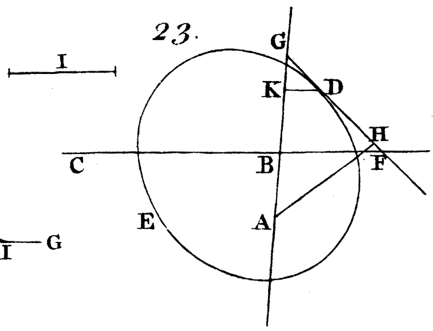
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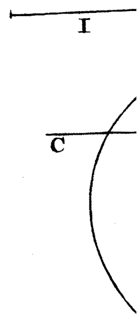
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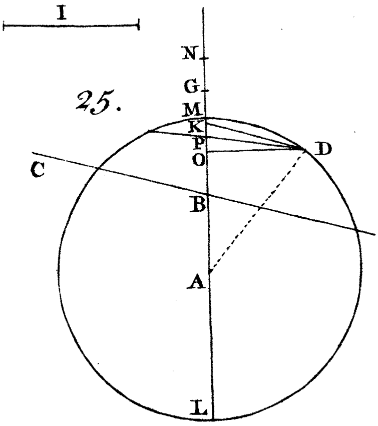
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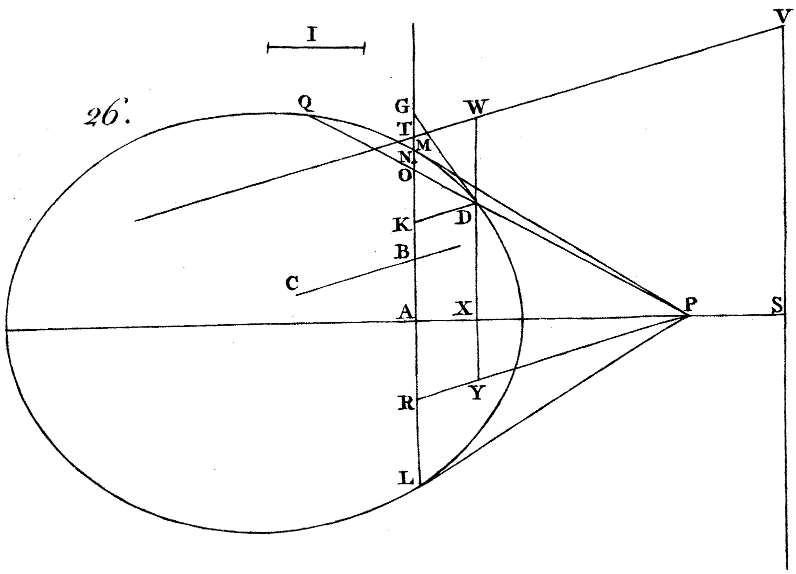
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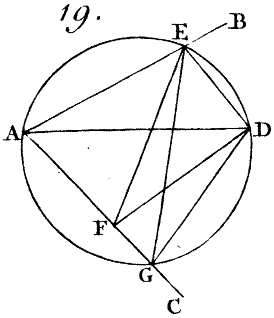

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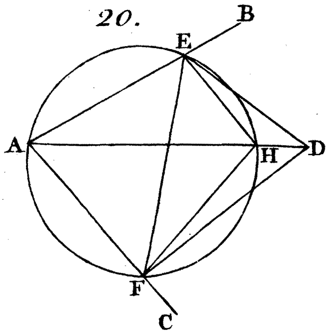

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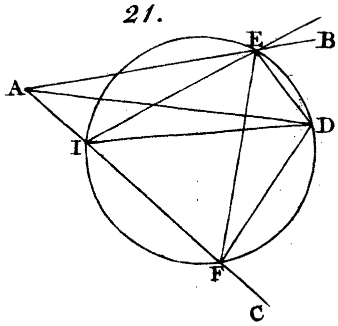

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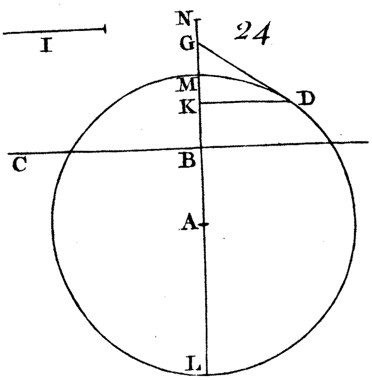

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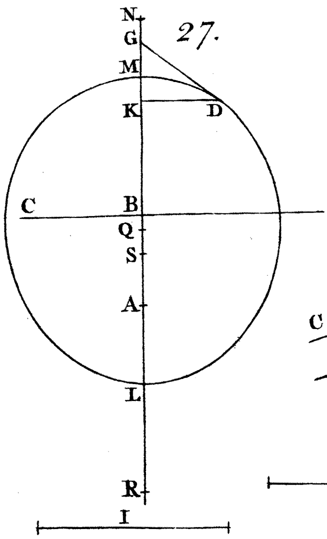

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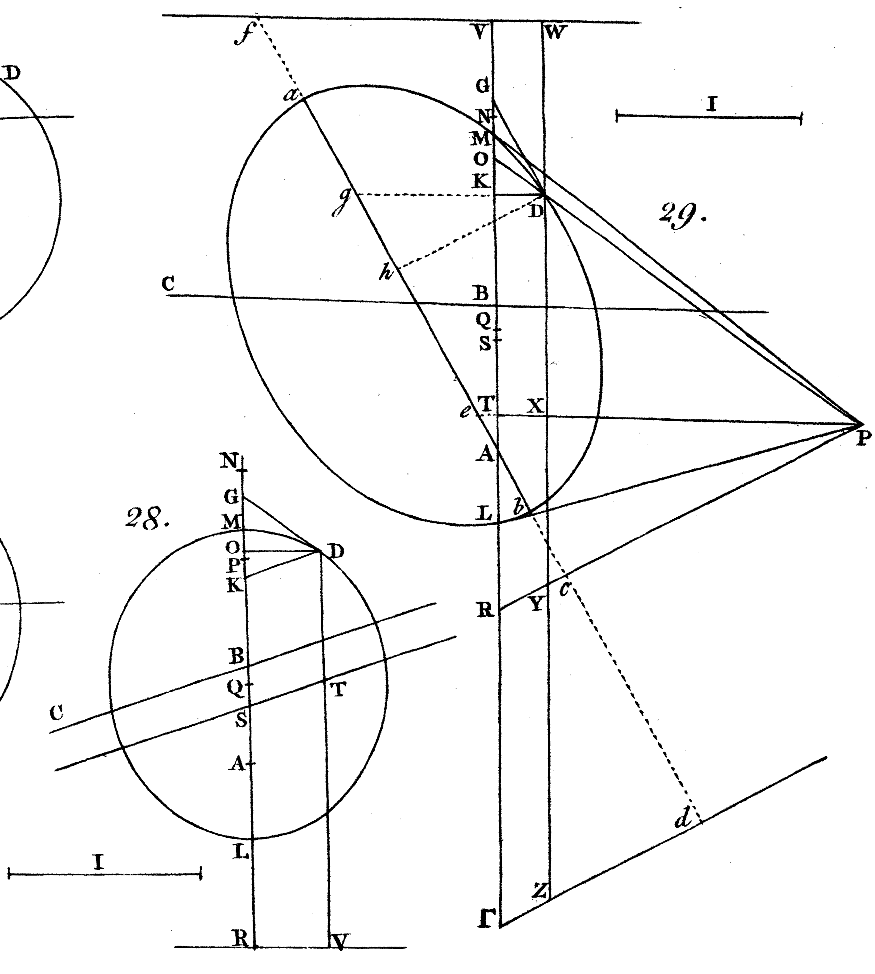

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THE describing a conic section through the angles of a quadrilateral with two parallel sides is so ready a means of assigning *loci* for the solution of solid problems, that it cannot be doubted, but this gave rise to the general problem concerning three and four lines mentioned by Apollonius, and described by Pappus; and it may be learnt from Sir Isaac Newton, who has considered the problem, how easily the most extensive form of it is reducible to the case, which I have supposed to give rise to it.

Sir Isaac Newton refers the general problem to this: Any quadrilateral A B C D being proposed, to find the *locus* of the point P, whereby PRQ being drawn parallel to AC and S P T parallel to A B, the ratio of the rectangle contained under QP, PR to that under SP, PT shall be given; and this by pursuing the steps, whereby he proves, that the point P will in every quadrilateral be in a conic section, may be readily reduced to the case of a quadrilateral with two sides parallel, after this manner. Draw B*t* and DN parallel to AC, then find the point M in ND, that the rectangle under MDN be to that under ANB in the ratio given, and draw C*r* M*d*.

TAB. XXIV. Fig. 1.

Here R*r* will be to AQ, or SP, as MD to AN, and B*t*, or QP, to T*t* as ND to NB whence the rectangle under R*r*, QP will be to that under SP, T*t* as that under MDN to that under ANB, that is, in the ratio given of the rectangle under RPQ to that under SPT. Therefore, by taking the sum of the antecedents and of the consequents,

consequents, the rectangle under r P Q will be to that under S P t, that is, to the rectangle under A Q B, in the quadrilateral A B C d, whose two sides A C, B d, are parallel, in the given ratio.

In like manner, if three of the given lines passed through one point, as the lines C A, C B, C D, and the rectangle under Q P R be to that under S P T in a given ratio, this case is with the same facility reduced to the like quadrilateral thus.

Fig. 2.

Draw B E parallel to A C, that shall cut S T produced in t, and let the point F be taken, that the rectangle under C A, E F be to the square of A B in the ratio given; then C r F being drawn, B t, or Q P, will be to T t as A C to A B, and R r to A Q, or S P, as E F to A B; whence the rectangle under Q P, R r will be to that under T t, S P, as that under A C, E F to the square of A B, that is, in the given ratio of the rectangle under Q P R to that under S P T, and the rectangle under Q P r will be to that under S P t or A Q B in the quadrilateral A B C F, whose two sides A C, B F are parallel, in the same given ratio.

Now let A B C D be a quadrilateral having the two sides A C, B D parallel, with any conic section passing through the four points A, B, C, D; also, the point E being taken in the section, and E F G being drawn parallel to A C or B D, let the ratio of the rectangle under A G B to the rectangle under F E G be given: then the conic section will be given.

Fig. 3, 4, 5.

Let the sides A B, C D meet in M, and draw M I bisecting A C and B D in K and L. Then the diameter

diameter of the ſection, to which A C and B D are lines ordinately applied, will be in the line M I; and if N P, Q S are tangents to the ſection, and parallel to A C and B D, the points O, R, in which they interſect M I, will be the points of their contact, and the vertexes of that diameter. But the ſquare of N O is to the rectangle under A N B, and the ſquare of Q R to the rectangle under A Q B, as the rectangle under E G H or F E G to that under A G B, therefore in a given ratio; but the ratio of N M to N O, the ſame as that of Q M to Q R, is alſo given; whence the ratio of the ſquare of N M to the rectangle under A N B, or of the ſquare of O M to the rectangle under K O L, is given, as likewiſe the ratio of the ſquare of R M to the rectangle under K R L. Fig. 3, 4.

Now in the ellipſis the ſquare of M O, the diſtance of the remoter vertex of the diameter O R from M, is greater than the rectangle under K O L; that is, the ratio given of the rectangle under F E G to that under A G B muſt be greater than the ratio of the ſquare of half the difference between A C and B D to the ſquare of A B. But in the hyperbola the ſquare of M O is leſs than the rectangle under K O L; whereby the ratio of the rectangle under F E G to that under A G B ſhall be leſs than that of the ſquare of half the difference between A C and B D to the ſquare of A B [a]. In

Fig. 3.

Fig. 4.

[a] As the ſquare of O M ſhall be greater or leſs than the rectangle under K O L, the ſquare of N M will be reſpectively greater or leſs than the rectangle under A N B; therefore the ratio of the ſquare of N O to the rectangle under A N B, that is, of the rectangle under F E G to that under A G B, will be accordingly greater

In both cafes, if the point T be fuch, that the rectangle under M O T be equal to that under L O K, whereby M O fhall be to O T in the given ratio of the fquare of M O to the rectangle under L O K, the given rectangle under K M L will be to the rectangle under L T K (by Prop. 35. L. 7. Papp. [b]) in this given ratio, and therefore given; confequently the points T and O will be given.

Fig. 3, 4.

In like manner, if the rectangle under MRV be equal to that under L R K, fo that M R be to R V in the given ratio of the fquare of R M to the rectangle under L R K, the given rectangle under K M L (by Prop. 22. L. 7. Papp.) will be to the rectangle under L V K in the fame given proportion, whence the points V and R will be given.

Thus in both cafes the points T and V will be found by applying to the given line K L a rectangle exceeding by a fquare, to which the given rectangle under K M L fhall be in the given ratio of the fquare of M O to the rectangle under K O L, or of the fquare of M R to the rectangle under K R L; M O being to O T, and M R to R V, in that given ratio.

But in the laft place, if this given ratio be that of equality, fo that the fquare of RM be equal to the rectangle under K R L, by adding to both the rectangle under MRL, that under RML will be equal to that under KM, LR, and MR to RL as KM to ML, and the vertex R of the diameter

Fig. 5.

greater or lefs than the ratio of the fquare of N O to the fquare of N M, which is the fame with that of the fquare of the difference between A K, B L to the fquare of A B.

[b] See pag. 511.

R I will be given, the conic section being here a parabola, this diameter having thus but one vertex.

Hitherto the point E, when the line E F G falls between A C and B D, is without the quadrilateral, and within the lines A B, C D, when E F G is without the quadrilateral.

But when E is within the lines A C, B D in the first case, and without in the second, the *locus* of the point E will be opposite sections, each passing through two angles of the quadrilateral.

When one section passes through A and C, and the other through B and D; then if the diameter M I be drawn, as before, and to K L be applied a rectangle deficient by a square, to which the given rectangle under K M L shall be in the given ratio of the square of M O to the rectangle under K O L, or of the square of M R to the rectangle under K R L, the points T and V, constituting the rectangles under K T L and under K V L, being thus found, M O will be to O T, and M R to R V, in this given ratio (by prop. 30. L. 7. Papp.) O and T being the vertexes of the diameter M I. Fig. 6.

But the rectangles under K T L, K V L cannot be assigned, as here required, unless the ratio given for that of the square of O M to the rectangle under K O L, or that of the square of R M to the rectangle under K R L, be not less than that of the rectangle under K M L to the square of half K L; that is, when the ratio of the square of O N to the rectangle under A N B, and that of the square of R Q to the rectangle under A Q B, or that of the given ratio of the rectangle under F E G to that under A G B is not less than that of the rectangle

 under

under AK, BL to the ſquare of half AB, or of the rectangle under AC, BD to the ſquare of AB.

But if one of the oppoſite ſections paſs through A and B, and the other through C and D, the ratio of the rectangle under FEG to that under AGB will be leſs than that of the rectangle under AC, BD to the ſquare of AB. For CL being drawn parallel to AB, and AD joined and continued to M, the line DM falls wholly within the ſection paſſing through C and D: therefore KM is leſs than KL, and the ratio of KD to KL leſs than that of KD to KM, that is, of BD to AB; whence BK being equal to AC, and CK to AB, the ratio of the rectangle under BKD to that under CKL, being the ratio of the rectangle under EGH, or FEG, to that under AGB, will be leſs than the ratio of the rectangle under AC, BD to the ſquare of AB.

Fig. 7.

And here the point L is given; for the given rectangle under BKD is to that under CKL in the given ratio of the rectangle under HGE, or that under FEG, to the rectangle under AGB; hence CK, equal to AB, being given, KL is given, and conſequently the point L.

Again, BL being joined, and NEOP drawn parallel to AB, alſo GEF continued to Q, as AG, equal to CQ, to FQ ſo will CK be to DK, and OP to EG, equal to OB, as KL to BK, conſequently the rectangle under OP, AG will be to that under EG, FQ as that under KL, CK to that under KB, DK, that is, as the rectangle under AGB to that under FEG; and by combining the antecedents and conſequents the rectangle under PEN will be to that under QEG in the ſame given ratio.

More-

Moreover D K being to A C as K M to C M, the ratio of D K to A C, that is, the ratio of the rectangle under BKD to the ſquare of AC, will be leſs than the ratio of K L to C L, or the ratio of the rectangle under C K L to that under A B, C L; therefore, by permutation and inverſion, the ratio of the rectangle under C K L to the rectangle under B K D, that is, the given ratio of the rectangle under N E P to that under A N C, equal to that under G E Q, is greater than the ratio of that under A B, C L to the ſquare of A C. And hence the oppoſite ſections paſſing through the angles of the quadrilateral A B C L, whoſe ſides A B, C L are parallel, will be given as before.

When the given ratio of the ſquare of O M to the rectangle under L O K ſhall be that of the rectangle under K M L to the ſquare of half K L, whereby the given ratio of the rectangle under F E G to that under A G B ſhall be that of the rectangle under A C, B D to the ſquare of A B, the points T and V ſhall unite in one, biſecting K L, and the points O and R ſhall alſo unite in one, dividing the line KLM harmonically; and then the *locus* of the point E will be each of the diagonals of the quadrilateral. Fig. 6.

In the laſt place, if the diagonals A D, B C of the quadrilateral were drawn, cutting G E in I and K, and the ratio of the rectangle under K E I to that under A I D were given, and not that of the rectangle under G E F to that under A G B; then the interſection of theſe diagonals, as L, will be in the line drawn from M biſecting A C, and B D, and the point L will fall within the quadrilateral, whereby the *locus*, when an Fig. 8.

 ellipſis

ellipsis or single hyperbola, will be assigned by the 36th proposition of the foresaid book of Pappus: and when opposite sections, by the 30th proposition, or be reduced to the preceding cases thus.

Since K G will be to G B as C A to A B, and I G to G A as B D to A B, the rectangle under K G I will be to that under A G B, in the given ratio of the rectangle under A C, B D to the square of A B. Therefore when the ratio of the rectangle under K E I to that under A I D is given, the rectangle under A I D also bearing a given ratio to that under A G B, the ratio of the rectangle under K E I to that under A G B will be given, and in the last place the ratio of the rectangle under G E F to that under A G B will be given, this rectangle under G E F being the excess of that under K G I above that under K E I [*c*]. And thereby the sections will be determined, as before.

AND thus may the *locus* of the point sought be assigned in all the cases of this ancient problem, which Sir Isaac Newton has distinctly explained. The other cases, he has alluded to, may be treated, as follows.

When three of the given lines shall be parallel, as A C, B D, and H I, the fourth line being A B, and K E L M being parallel to A B, the ratio of the rectangle under K E L to the rectangle under E G and E M shall be given; that is, three points A, B, and H being given in the line A B, with the line G E insisting on A B in a given angle, that the rectangle under AGB shall be to that under G H and G E in a given ratio: then

Fig. 9.

[*c*] By Prop. 193. Lib. 7. Papp.

take

take A N equal to B H, and draw N O parallel to A C, B D, and H I.

Then if N P be drawn, that P O be to O N in the given ratio, N P will be given in poſition, and P O will be to O N, that is, E G, as the rectangle under K E L to that under E M, E G; ſo that the rectangle under K E L will be equal to that under PO, EM. But the rectangle under O K M is equal to the exceſs of that under O E M above that under K E L [d]; therefore the rectangle under O K M, or that under N A H, or under N B H, is equal to that under E M and the exceſs of O E above O P, that is, to the rectangle under P E M; the point E therefore is in an hyperbola deſcribed to the given aſymptotes PN, MH, and paſſing through A and B.

Again if two of the given lines only are parallel, but the rectangles otherwiſe related to them, than as above. Suppoſe the ratio of the rectangle under A G, E F to that under B G, G E is given. Let C D meet A B in L, and let H E I, M F N be drawn parallel to A B, and L K parallel to A C and B D. Then the parallelogram E M will be to the parallelogram E B in the given ratio. Fig. 10. Take A O to O B in that ratio, and draw O P parallel to A C and B D. Here the point O will be given, and the parallelogram P A will be in the given ratio to the parallelogram P B; whence A B will be to B O as the parallelogram B H to the parallelogram B P, and as the difference between the parallelograms E M and E B to the parallelogram E B, conſequently as the parallelogram G M to the parallelogram P G; therefore the ratio of the rectangle under A G, F G to the rectangle under

[d] By Prop. 194. Lib. 7. Papp.

E G,

E G, E P or O G will be given; and in the last place the ratio of F G to G L being given, the ratio of the rectangle under A G and G L to that under E G, O G will be given. And thus three points A, L, O, will be given with G E insisting on A B in a given angle, as in the preceding case.

Moreover, AC and BD being parallel, AB and CD may be also parallel. And then, when the ratio of the rectangle under A G B to that under G E F is given, the determination of the *locus* is so obvious as not to have required a distinct explanation. But when the rectangle under A G, E F bears a given ratio to that under B G, G E; let the diagonals A D, B C be drawn, and H E L K drawn parallel to A D. Then the rectangle under H E L will be to that under K E I in the same given ratio; and if C M be taken to M B in the same ratio, the lines M N P, M O Q drawn, the first parallel to A C, B D, and the other parallel to A B, C D, will be given in position, and the diagonal B M will bisect both I K, N O, and H L; therefore the rectangle under H E L being to that under K E I as M C to M B, that is, as N H to N K, here by division the rectangle under H E L will be to that under I H K [e] as NH to HK; therefore equal to that under N H and I H or K L. But the rectangle under N E O is equal to the sum of the rectangles under H N L and under H E L [f]; therefore the rectangle under N E O is equal to that under N H, N K, equal to that under A P D, that is, equal to that under P A Q, or that under P D Q, the diagonal B M bisecting both P Q and

Fig. 11.

[e] By the prop. of Papp. before cited. [f] By the same.

A D.

A D. But thus the point E is in an hyperbola deſcribed to the aſymptotes M N, MO, and paſſing through A and D.

THE determination of this *locus* for three lines is ſolved almoſt explicitly by Apollonius in the three laſt propoſitions of his third book of Conics. For if the three lines propoſed were A B, A C, B C, and the point ſought D, that the ratio of the rectangle under E D F (the line E F being drawn parallel to B C) ſhould be in a given ratio to the ſquare of a line drawn from D to B C in a given angle, the ſquare of which line will be in a given ratio to the rectangle under B E, C F; then if B H, C I are drawn parallel to A C and A B reſpectively, alſo B D L, C D K drawn through D, the ſquare of B C will be to the rectangle under B K, C L as the rectangle under D F, D E, to that under C F, B E. Fig. 12, 13, 14.

Hence if the ratio of the rectangle under D F, D E to the ſquare of a line drawn from D on B C in a given angle, is given; the ſquare of this line being in a given ratio to the rectangle under C F, B E, the ratio of the rectangle under B K, C L to the ſquare of B C will be given; whence a conic ſection paſſing through D will in all caſes be given.

In the firſt place let the point D be within the angle B A C. Then if B C be biſected by the line A M, this will be a diameter to the conic ſection, which ſhall touch B A, A C in the points B, C, and B C will be ordinately applied to that diameter; the vertex of this diameter being N, the given ratio of the rectangle under B K, C L to Fig. 12.

2 the

the ſquare of B C will be compounded of the ratio of the ſquare of M N to the ſquare of NA, and of the ratio of the rectangle under B A C to the fourth part of the ſquare of B C; and thus the line A M will be divided in N in a given ratio, and the point N, one vertex of the diameter, to which B C is ordinately applied, will be given.

If A N be equal to N M, the point N will be the only vertex of this diameter, and the ſection will be a parabola.

Otherwiſe by taking the point O in A M extended, ſo that the ratio of A O to O M be the ſame with that of A N to N M, the point O will be the other vertex of the diameter.

And here if the ratio of A N to N M be that of a greater to a leſs, the point O will fall beyond M from A within the angle B A C, the conic ſection being an ellipſis.

But if the ratio of A N to N M be that of a leſs to a greater, the point O will fall on the other ſide of A, and the ſection will be an hyperbola.

Fig. 13. And in this caſe if the oppoſite ſection be drawn, that alſo will be the *locus* of the point D within the angle vertical to the angle B A C.

In the laſt place, if D be in either of the collateral angles, A M drawn as before will contain a ſecondary diameter in oppoſite ſections, one of which shall touch B A in B, and the other C A in C. Then if one of theſe ſections paſs thro' D, the ſections will be given. For here P A Q being drawn through A parallel to B C, the given ratio of the rectangle under C L, B K to the ſquare

Fig. 14.

of

of B C will be the ſame with that of the given rectangle under B A C to the ſquare of A P: therefore A P is given, and thence the ſections. For let R S be the ſecondary diameter, to which B C is ordinately applied, and T the center of the oppoſite ſections. Then the ſquare of B M will be to the rectangle under A M T as the ſquare of the tranſverſe diameter conjugate to the ſecondary diameter R S to the ſquare of this ſecondary diameter; and if a line were drawn from M to P, this would touch the hyperbola B P in P [g], and the ſquare of A P will be to the rectangle under M A T in the ſame ratio; therefore the given ratio of the ſquare of M B to the ſquare of A P will be that of the rectangle under A M T to the rectangle under M A T, or the ratio of M T to A T; conſequently the ratio of M T to A T is given, and thence the point T. But alſo the diameter R S is given in magnitude, the ſquare of R T or of S T being equal to the rectangle under M T A; whence in the laſt place the tranſverſe diameter conjugate to this is alſo given; for the ſquare of this diameter is to the ſquare of R T as the given ſquare of B M to the rectangle under A M T now alſo given.

But a more ſimple caſe may alſo be propoſed in three lines, when the ratio of the rectangle under E D F ſhould be equal to the rectangle under a given line, and that drawn from D to B C in a given angle. Fig. 15.

This line will bear, both to B E and F C, a given ratio, and the rectangle under EDF will be in a given

[g] Apoll. conic. L. II. prop. 40.

ratio to the rectangle under the given line and E B or C F.

Let the line given be H, and take M B and N C, that the rectangle under M B C, and that under B C N be to that under B A and H in the given ratio of the rectangle under E D F to that under B E and H, B M and C N being equal. Then draw from M and N lines parallel to B A, C A, which shall intersect E F in K and L, whereby, M K cutting C A in I, the rectangle under M B C will be to that under B A and H as the rectangle under B M C to that under M I and H, and also as the rectangle under E K F to that under K I and H, that is, as the rectangle under E D F to that under H and B E or M K, whence by adding the antecedents and consequents the rectangle under K D L will be to the rect-tangle under H and M I in the same given ratio, which is also that of the rectangle under B M C to the same rectangle under H and M I: the point D therefore is in an hyperbola passing through B and C having for asymptotes the lines M K and N L given in position, the rectangle under K D L being equal to that under B M C, or that under M B N.

If the two lines A B and A C are parallel, the *locus* may be known to be a parabola by the last proposition of the fourth book of Pappus.

But if B C were parallel to one of the other, the *locus* will be an hyperbola, as the preceding, but assigned by a shorter process.

Fig. 16. Suppose the given lines to be A E, A F, and B C parallel to A F. And let the rectangle under E D F be equal to that under D G, and the given line H, the line E G making given angles with A E, A F. Here take E I equal to H,

 and

and deduct from both the rectangles that under E I or H, and DF, whereby will be left the rectangle under IDF equal to that under H and FG, both whose sides are given. Draw therefore IK parallel to A E, and the rectangle under I D F will be equal to this given rectangle, the given lines KI, AF being the asymptotes to the hyperbola passing through D.

Coroll. If L M be drawn through B parallel to E F, L B shall be equal to F G, and B M equal to E I or H, whereby the hyperbola opposite to that passing through D will pass through B.

SCHOLIUM.

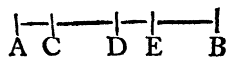
The propositions of Pappus, which have been referred to in pag. 500, 501, 504, l. 2. are given by him, among others, for Lemmas subservient to the lost treatise of Apollonius *De sectione determinata*, and the four here cited respect and comprehend all the cases of the problem, where three points are given in any line, and a fourth is required such, that the rectangle under the segments of the proposed line intercepted between the point sought, and two of the given points, shall bear a given ratio to the square of the segment terminated by the third point.

The cases indeed of the problem, from the diversity of situation in the points given to the point sought and to one another, are in number six. The given extreme of the segment to constitute the square may either be without the other two given points, or between them. And when it is without, the point sought may be required to be taken without them all, either on the side opposite to the given extreme of the segment to constitute the square, which will be one case, or it may be required to fall on the same side,

 which

which will be a ſecond caſe. If it be required to fall between this point and the other two, this will be a third caſe. A fourth caſe will be, when the point ſought ſhall be required to fall between the other two points. Alſo when the given extreme of the ſegment to conſtitute the ſquare lies between the other two given points, the point ſought may be required to fall, either there alſo, or without, compoſing the 5th, and 6th caſes.

The propoſitions in Pappus referring to theſe caſes, though but four in number, ſuffice for them all, each propoſition being applicable to the problem two ways. For inſtance the thirty-fifth propoſition, as expreſſed by Pappus, is this, being the firſt above cited. Three points C, D, E being taken in the line A B, ſo that the rectangle under ABE be equal to that under CBD, AB is to BE as the rectangle under D A C to that under C E D. Now A B is to B E, both as the ſquare of A B to the rectangle under A B E, and as the rectangle under A B E to the ſquare of B E. Therefore if the ratio of A B to B E be given, the ratio of the ſquare of A B to the rectangle under C B D will be given, which is the firſt of the caſes above deſcribed, and alſo the ratio of the rectangle under C B D to the ſquare of B E given, which is the ſecond caſe. In both caſes the rectangle under D A C will be to that under C E D in the given ratio of A B to B E. But in the firſt the rectangle under D A C will be given, and the point E in the rectangle under C E D to be found by applying a rectangle, which ſhall bear a given ratio to the given rectangle under D A C to the given line C D exceeding by a ſquare; and in the ſecond caſe the rectangle

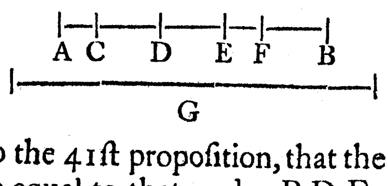


rectangle under C E D is given, and A in the rectangle under D A C to be found by applying to the given line C D a rectangle exceeding by a ſquare, which ſhall bear a given ratio to the rectangle under C E D now given; whence by the ratio of A B to B E given the point B will be found in both caſes.

The 22d propoſition either way applied refers to the 3d caſe only, the 30th relates both to the 4th and 5th, and the 36th propoſition to the remaining 6th.

The 45th, and other following propoſitions, are accommodated to the ſolution of Apollonius's problem, when four points are given, and a fifth required, which with the given points ſhall form four ſegments ſuch, that the rectangle under two ſhall bear a given proportion to the rectangle under the other two. The various caſes of this problem appear to have been the ſubject of the ſecond book of the mentioned treatiſe of Apollonius; and, according to the character given by Pappus of thoſe propoſitions, theſe lemmas ſerve to reduce them to problems in the firſt book, not thoſe above mentioned, but thoſe, where three points being given, the rectangle under the ſegments included by two, and a fourth point ſhall bear a given proportion to the rectangle under the ſegment formed by the third point and a given line.

For inſtance the 46th propoſition is this; in the line A B four points A, C, E, B being given; and the point F aſſumed between E and B;

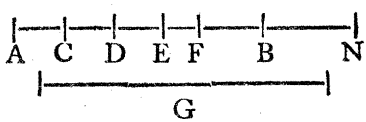


alſo D taken, according to the 41ſt propoſition, that the rectangle under A D C be equal to that under B D E; if G be equal to the ſum of A E, C B, the rectangle

gle under A F C together with that under E F B will be equal to the rectangle under G and D F.

Here if it were propoſed to find the point F, that the ratio of the rectangle under A F C to that under E F B ſhould be given, the ratio of the rectangle under A F C to that under DF and the given line G would be given.

But this analyſis may be carried on to a compleat ſolution of the problem thus. If C N be taken to G in the given ratio of the rectangle under A F C to that under D F and G, the point N will be given, and the rectangle under A F, C N will be to



that under A F, G in this ratio of CN to G; conſequently the exceſs of the rectangle under A F, C N above that under A F C, that is, the rectangle under A F N, will be to the exceſs of the rectangle under A F and G above that under D F and G, or the given rectangle under A D, G, in the ſame given ratio, and in the laſt place the rectangle under A F N will equal the given rectangle under A D and C N.

Here I have choſen this propoſition in particular, becauſe the caſe of the problem, to which it is ſubſervient, is ſubject to a determination, when F N ſhall be equal to A F. And then the rectangle under AFN being equal to that under AD and CN, as CN to FN ſo is AF to AD, and by diviſion as CF to FN ſo DF to AD; therefore when AF is equal to FN, CF will be to A F as F D to A D: conſequently C D to F D as F D to AD, and the ſquare of D F equal to the rectangle under A D C, when the problem admits of a ſingle ſolution only, wherein the rectangle under

A F C

AFC will bear to that under EFB a leſs ratio than in any other ſituation of the point F between E and B.

Moreover CN is to G as the rectangle under AFC to the ſum of the rectangles under AFC and EFB; therefore FN being equal to AF, when the problem is limited to this ſingle ſolution, the rectangle under AFC ſhall be to the rectangles under AFC and EFB together as the ſum of AF and FC to G, which is equal to the ſum of AE and CB; whence by diviſion the ratio of the rectangle under AFC to that under EFB, when the problem is limited to this ſingle ſolution, will be that of the ſum of AF and CF to the exceſs of FB above EF.

Thus directly do theſe lemmas correſpond with Apollonius's firſt mode of ſolution, and lead to the general principle of applying to a given line a rectangle exceeding or deficient by a ſquare, which ſhall be equal to a ſpace given. This being a ſimple caſe of the 28th and 29th propoſitions of the 6th book of Euclid's elements, admits of a compendious ſolution. Such a one is exhibited by Snellius in his treatiſe on theſe problems (in Apollon. Batav.) and Des Cartes has exhibited another more contracted in it's terms, but not therefore more uſeful. It may alſo be performed thus. If upon a given line AB any triangle ACB be erected at pleaſure; then if the legs CA, CB, whether equal or unequal, be continued to D and E, that the rectangles under CAD and CBE be each equal to the given ſpace, and a circle be deſcribed through C, D, E cutting AB extended in F and G, the rectangle under BFA and BGA will each be equal to the ſpace given. Alſo if in the legs CA, CB the rectangles under CAD and CBE be each

Fig. 17.

Fig. 18.

taken

taken equal to the ſpace given, and a circle in like manner be deſcribed through C, D, E, cutting AB in F and G, the rectangles under AFB and AGB will each be equal to the given ſpace. Here it is evident, that the ſpace given muſt not exceed the ſquare of half AB, when equal, the circle will touch AB in it's middle point.

POSTSCRIPT.

AS this application to a given line of a rectangle exceeding or deficient by a ſquare, or the more general problem treated of in the ſixth book of the elements, of applying a ſpace to a line ſo as to exceed or be deficient by a parallelogram given in ſpecies, is the moſt obvious reſult, to which the analyſis of plane problems, not too ſimple to require this conſtruction, leads; ſo the deſcriptions of the conic ſections here treated of, ſtand in the like ſtead in regard to the higher order of problems ſtyled ſolid from the uſe of the conic ſections deemed neceſſary for their genuine ſolution. And theſe are the only modes of ſolution, the modern algebra, which grounds its operations on one or two elementary propoſitions only, naturally leads to. But as the form of analyſis amongſt the antients, by expatiating through a larger field, often was found to arrive at concluſions much more conciſe and elegant, than could offer themſelves in a more confined track; the antient ſages in geometry, that the ſolid order of problems might not want this advantage, ſought out that copious and judicious collection of properties attending the conic ſections, which

which, with ſome uſeful additions from later writers, have been handed down to us.

And as the advantages of this ancient ſyſtem of analyſis cannot be too much inculcated in an age, wherein it has been ſo little known, and almoſt totally neglected, permit me, Sir, to cloſe this addreſs to you with an example in each ſpecies of problems.

Were it propoſed to draw a triangle given in ſpecies, that two of its angles might touch each a right line given in poſition, and the third angle a given point. It is obvious, how difficult it would be to adopt a commodious algebraic calculation to this problem; notwithſtanding it admits of more than one very conciſe ſolution, as follows.

Let the lines given in poſition be AB, AC and the given point D, the triangle given in ſpecies being EDF. Fig. 19, 20, 21.

In the firſt place ſuppoſe a circle to paſs through the three points A, E, D, which ſhall interſect AC in G. Then EG, DG being joined, the angle DEG will be equal to the given angle DAC, both inſiſting on the ſame arch DG; alſo the angle EDG is the complement to two right of the given angle BAC: theſe angles therefore are given, and the whole figure EFGD given in ſpecies. Conſequently the angle EGF, and its equal ADE will be given together with the ſide DE of the triangle in poſition. Fig. 19.

Again, ſuppoſe a circle to paſs through the three points A, E, F, cutting AD in H, and EH, FH joined. Here the angle EFH will be equal to the given angle EAH, and the an- Fig. 20.

gle F E H equal to the given angle F A H. Therefore the whole figure E H F D is given in ſpecies, and conſequently the angle A D E, as before.

In the laſt place ſuppoſe a circle to circumſcribe the triangle, and interſect one of the lines, as A C, in I. Here D I being drawn, the angle D I F will be equal to the given angle D E F in the triangle; conſequently D I is inclined to A C in a given angle, and is given in poſition, as alſo the point I given; whence I E being drawn, the angle F I E will be the complement of the angle E D F in the triangle to two right. Therefore I E is given in poſition, and by its interſection with the line A B gives the point E, with the poſition of D E, and thence the whole triangle, as before.

Fig. 21.

Here it may be obſerved, that the angle D of the triangle E D F given in ſpecies touching a given point D, and another of its angles touching A C, the line I E here found is the locus of the third angle E.

Again, in the aſtronomical lectures of Dr. Keil, it is propoſed to find the place of the earth in the ecliptic, whence a planet in any given point of its orbit ſhall appear ſtationary in longitude, and a ſolution is given from the late eminent aſtronomer, Dr. Halley, upon the aſſumption, that the orbit of the earth be conſidered as a circle concentric to the Sun.

But for a compleat ſolution of this problem let the following lemma be premiſed.

The velocity of a planet in longitude bears to the velocity of the earth the ratio, which is compounded of the ſubduplicate ratio of the *latus rectum* of the greater axis of the planet's orbit to the *latus rectum* of of the greater axis of the earth's orbit, of the ratio of the cosine of the angle, which the orbit of the planet makes with the plane of the ecliptic, to the radius, and of the ratio of a line drawn in any angle from the center of the sun to the tangent of the orbit of the earth at the point, wherein the earth is, to a line drawn in the same angle from the sun to the tangent of the orbit of the planet projected upon the plane of the ecliptic at the place of the planet in the ecliptic.

Let A be the sun, BC the orbit of any planet, DE the same projected on the plane of the ecliptic, FG being the line of the nodes, B the place of the planet in its orbit, D its projected place: then the plane through B and D, which shall be perpendicular to both the planes BC and DE, intersecting those planes in BH, DH, the lines BH, DH will be both perpendicular to the line of the nodes, and the angle BHD the inclination of the orbit to the plane of the ecliptic. But tangents drawn to BC and DE at the points B and D respectively will meet the line of the nodes, and each other in the same point I, and the velocity of the planet in longitude will be to its velocity in the orbit BC, as DI to BI.

Now from the point A let AK fall perpendicular on BI, and AL be perpendicular to DI: then the ratio of DI to IB will be compounded of the ratio of DI to DH, or of AI to AL, of the ratio of DH to BH, and of that of BH to BI, that is, of AK to AI. But DH is to BH as the cosine of the inclination of the orbit to the radius, and the two ratios, that of AI to AL, and that of AK to AI, compound the ratio of AK to AL: therefore

 the

the velocity of the planet in longitude is to the velocity in its orbit in the ratio compounded of that of the cofine of the inclination of the planet's orbit to the radius, and that of A K to A L.

Moreover the ratio of the velocity of the planet in B to the velocity of the earth in any point of its orbit is compounded of the fubduplicate of the ratio of the *latus rectum* of the greater axis of the planet's orbit to the *latus rectum* of the greater axis of the earth's orbit, and of the ratio of the perpendicular let fall from the fun on the tangent of the earth's orbit at the earth to A K, the perpendicular let fall on the tangent of the planet's orbit at B. Therefore the velocity of the planet in longitude, when in B, to the velocity of the earth in any point of it's orbit is compounded of the fubduplicate ratio of the *latus rectum* of the greater axis of the planet's orbit, to the *latus rectum* of the greater axis of the earth's orbit, of the ratio of the co-fine of the inclination of the planet's orbit to the radius, and of the ratio of the forefaid perpendicular on the tangent of the earth's orbit to A L, the perpendicular on D I: thefe perpendiculars being in the fame ratio with any lines drawn in equal angles to the refpective tangents.

This being premifed, the place of a planet in the ecliptic being given, the place of the earth, whence the planet would appear ftationary in longitude, may be affigned thus.

A denoting the fun, let B be a given place of any planet in it's orbit projected orthographically on the plane of the ecliptic, C B the tangent to the planet's projected orbit at the point B, which will therefore be given in pofition. Alfo let D E be

Fig. 23.

the

the orbit of the earth, and the point D the place of the earth, whence the planet would appear ſtationary in longitude at B.

Join AB, and draw a tangent to the earth's orbit at the point D, which may meet CB in F, and the line AB in G; draw alſo AH making with DF the angle AHD equal to that under ABC. Then the point D being the place, whence the planet appears ſtationary in longitude, as FB to FD ſo will the velocity of the planet in longitude in B be to the velocity of the earth in D, this velocity of the planet in B being alſo to the velocity of the earth in D in the ratio compounded of the ſubduplicate of the ratio of the *latus rectum* of the greater axis of the planet's orbit, to the *latus rectum* of the greater axis of the orbit of the earth, of the ratio of the co-ſine of the inclination of the planet's orbit to the plane of the ecliptic to the radius, and of the ratio of AH to AB: therefore the ratio of FB to FD will be compounded of the ſame ratios; and if I be taken, that the ratio of AB to I be compounded of the two firſt of theſe, I will be given in magnitude, and the ratio of FB to FD will be compounded of the ratio of AB to I, and of AH to AB. Whence FB will be to FD as AH to I; and the angles CBA, or FBG, and AHG being equal, whereby FG will be to FB as AG to AH, by equality FG will be to FD as AG to I, and DK being drawn parallel to FB, BG will be to BK as FG to FD, and therefore as AG to I.

But now, as this problem may be diſtributed into various caſes, in the firſt place conſider the earth as moving in a circle concentric to the ſun, and likewiſe CB, the tangent to the planet's orbit, perpendicular to AB.

But

But here D K alſo will be perpendicular to A B, and A B meeting the earth's orbit in L and M, the rectangle under K A G will be equal to the ſquare of A M. But B G being to B K as A G to I, if B N be taken equal to I, B G will be to B K as A G to B N, and A B to K N alſo as A G to B N, and the rectangle under N K, A G equal to that under A B and I: therefore the rectangle under K A G being equal to the ſquare of A M, N K will be to K A as the rectangle under A B, I to the ſquare of A M, that is, in a given ratio, and K D with the point D will be given in poſition.

Fig. 24.

Again, when C B is not perpendicular to L M, let D O be perpendicular to L M. Then the rectangle under O A G will be equal to the ſquare of A M. But B N being taken equal to I, as before, the rectangle under N K, A G will be equal to that under A B, I; whence N K will be to A O in the given ratio of the rectangle under A B, I to the ſquare of A M, Therefore N P being taken to P A in that ratio, the point P will be given, and K P, the exceſs of N P above N K, will be to P O, the exceſs of A P above A O, in the ſame ratio. Hence, as D K is parallel to C B and D O perpendicular to L M, the triangle K O D is given in ſpecies, and if P D be drawn, the angle O P D will be given; for the co-tangent of the angle O K D will be to the co-tangent of the angle O P D, as K O to O P, that is, as the rectangle under A B, I together with the ſquare of A M to the ſquare of A M, and hence the point D is given by the line P D drawn from a given point P in a given angle A P D; and if A D be drawn, A D will be to A P as the ſine of the angle A P D to the ſine of the angle P D A; this angle therefore

Fig. 25.

therefore is given, and the angles A P D, P D A being given, the angle P A D is given.

Coroll. Here, where the orbit of the earth is ſuppoſed a circle, the ratio of I to A B, that is, of the rectangle under A B, I to the ſquare of A B, will be compounded of the ſubduplicate ratio of A M, the ſemidiameter of the earth's orbit, to half the *latus rectum* to the greater axis of the planet's orbit, and of the ratio of radius to the co-ſine of the inclination of the planet's orbit to the plane of the ecliptic; and adding on both ſides the ratio of the ſquare of A B to the ſquare of A M, the ratio of the rectangle under A B, I to the ſquare of A M will be compounded of the ratio of the ſquare of A B to the rectangle under A M and the mean proportional between A M and the half of this *latus rectum* of the planet's orbit, and of the ratio of the radius to the co-ſine of the inclination of the net's orbit.

In the next place, though the earth's orbit is not a circle concentric to the ſun; yet if the projection of the planet falls on the line perpendicular to the axis of the earth's orbit, the point A will ſtill biſect L M.

In this caſe draw to the points L and M tangents to the ellipſis meeting in P, from whence through D draw P D meeting the ellipſis again in Q, and interſecting L M in O. Here if a tangent be drawn to the ellipſis in Q, it will meet the tangent at D on the line L M in the point G. Fig. 26.

Now L G will be to G M as L O to O M, and the point A biſecting L M, the rectangle under G A O will be equal to the ſquare of A M. But B G is to B K as A G to I. Therefore B N being taken equal to I, A B will be to K N as A G to I, and the rectangle under A B, I equal to that under A G, KN: whence AO being to KN as the rectangle under GAO to that under AG and KN, AO will be to KN as the given ſquare of AM to the rectangle under AB and I, alſo given.

Draw RP parallel to CB, and take PS to AP, alſo NT to AR in this given ratio inverted. Then will the points T and S be both given, alſo AO will be to KN, and RO to KT, as AR to NT, that is, as AP to PS. Therefore if TV be drawn parallel to CB, that is, to KD, and VS parallel to LM, theſe lines will be both given in poſition; and WDXY being alſo drawn parallel to LM, WD will be equal to KT, and RO being to KT, as AP to PS, DY will be to WD as XP to PS, and by compoſition YW to WD as XS to PS, and the given rectangle under YW, or SV, and PS equal to that under WD, and XS. Whence SV being parallel to LM, the point D will be in an hyperbola paſſing thro' P, and having for aſymptotes the lines VS, VT given in poſition.

But if the projection of the planet fall on the axis of the earth's orbit, or the ſame continued, AB extended to the earth's orbit in L and M will be the axis of that orbit.

If alſo CB ſhould be perpendicular to AB, KD would be ordinately applied to LM; and the point R being taken, that Q being the center of the orbit, the rectangle under AQR be equal to the ſquare of QM, the ſame will be equal alſo to the rectangle under GQK; whence as GQ to AQ ſo RQ to QK, and AG to AQ as KR to QK. But, as above, BG being to BK as AG to I, and BN taken equal to I, BG will be to BK as AG to BN, and AB to KN alſo as AG to BN or I. Therefore if NS be taken to AB as I to AQ, by

Fig. 27.

equality

equality N S will be to N K as A G to A Q, that is, as K R to Q K; and in the laſt place N S to K S as K R to Q R, that is, the rectangle under S K R equal to the given rectangle under N S, Q R; whence the point K, the poſition of K D, and thence the point D will be given.

But if D K be not ordinately applied to L M, let D O be ordinately applied to L M. Then here the rectangle under A Q R, equal to the ſquare of Q M, will be equal to that under O Q G, and G Q to A Q as Q R to O Q, whence by compoſition A G to A Q as O R to O Q. But B N being now alſo taken equal to I, and N S to A B as I to A Q, A B will be here in like manner to K N as A G to I, and N S to K N as A G to A Q: therefore N S will be to K N as O R to O Q, and by converſion N S to K S as O R to Q R. But N S and Q R being both given in magnitude, if S P be taken to N S as Q R to P R, the point P will be given, and alſo by equality S P will be to K S as O R to P R; whence if R V be drawn parallel to D O, and S T to K D, both R V and S T will be given in poſition, one paſſing through the given point R, parallel to the ordinates applied to the axis L M, and the other through the point S alſo given, and parallel to K D or C B: alſo D T V being drawn parallel to M L, D T will be equal to K S and D V equal to O R, therefore as S P to D T ſo D V to P R, and the rectangle under S P R equal to that under T D V, conſequently the point D in an hyperbola paſſing thro' P, and having for aſymptotes the lines S T, R V given in poſition.

Fig. 28.

In

In the laſt place when the line L M drawn through the ſun in A, and the projected place of the planet in B, is neither the axis of the earth's orbit, nor biſected in A, the tangents to the points L, M being drawn to meet in P, let LM be biſected in Q, and the point R taken, that the rectangle under A Q R be equal to the ſquare of Q M, whereby P D O being drawn, the rectangle under A Q R ſhall be equal to that under O Q G, and Q G to A Q as Q R to Q O, or by compoſition A G to A Q as O R to Q O. Therefore if N B be here alſo taken equal to I, and N S to A B as I to A Q, A B being as before, to N K as A G to I; by equality N S will be to N K as A G to A Q, that is, as O R to Q O. Whence by converſion N S will be to K S as O R to Q R; and if P T be drawn parallel to C B and S V be here taken to N S as Q R to T R, by equality S V will be to K S as O R to T R and alſo by converſion S V to K V as O R to O T. Moreover S V will be given in magnitude, and the point V given; therefore V W drawn parallel to C B, or K D, will here be given in poſition. But W D X Y being alſo drawn parallel to R V, S V will be to K V, or D W, as Y D to X D, and Y Z being taken equal to the given line S V, Y Z will be to D W as Z D to X W, equal to T V, and the given rectangle under Y Z, T V equal that under W D Z. Therefore Γ Z being drawn parallel to R P, R Γ, and its equal Y Z, being given, the line Γ Z is given in poſition, and the point D in an hyperbola having for aſymptotes V W, Γ Z, and paſſing through P.

Fig. 29.

Thus is this problem in all cafes folved either by a right line, or an hyperbola given in pofition, which fhall interfect the projected orbit in the point fought. For though in each cafe the projection of the planet has here been confidered as within the orbit of the earth, the form of argumentation will be altogether fimilar, were the projection of the planet without. And this is agreeable to the method, I have purfued throughout this difcourfe, where I have always accommodated the expreffion to one fituation only of the terms given and fought in each article; the variation neceffary for the other cafes, when one has been duely explained, being fufficiently obvious.

In the 5th volume of the Commentaries of the Royal Academy at Peterfbourg is given an algebraical computation for a general folution of this problem in the orbits of any two planets projected on the plane of the ecliptic; but with this overfight of applying to the projected orbits a propofition from Dr. Keil's Aftronomical Lectures, which relates to the real orbits (*a*).

However from the geometrical folution now given a calculation for affigning the point D may be formed without difficulty. L D M being the orbit of the earth, A is the focus, and R P perpendicular to the

(*a*) The demonftration of Dr. Keil's propofition proceeds on the known property in the planets of having their periodic times in the fefquiplicate ratio of the axes of their orbits, which confines the propofition to the real orbits; for in each planet the periodic time through the projected orbit is the fame, as through the real, though the axis in one be not equal to the axis of the other.

 axis.

axis. Let this axis be *a b* meeting R P in *c*, Γ Z in *d*, P T in *e* and W V in *f*. Then the angle *a* A M is given, being the diſtance between the heliocentric place of the planet in the ecliptic from the earth's aphelion. Alſo P T being parallel to C B, the angle A T *e*, and conſequently the angle A *e* T, will in like manner be given, whence the points Γ, R, T, V being given, as in the ſolution above, the points *d*, *c*, *e*, and *f* will be given, the triangles A R *c*, A T *e*, being given in ſpecies, and ſimilar reſpectively to the triangles A Γ *d*, and A V *f*. Alſo the rectangle under W D Z being equal to that under R Γ, V T, if D K be continued to the axis in *g*, and D *h* be drawn parallel to P R, the rectangle under *f g*, *h d* is equal to that under *f e*, *d c*, and both being deducted from the rectangle under *f h d* the exceſs of the rectangle under *f h d* above that under *f e*, *d c* will be equal to that under *g h d*, ſo that this difference will be a mean proportional between the ſquare of *h d* and the ſquare of *h g*, which is in a given ratio to the ſquare of *h* D, and therefore in a given ratio to the rectangle under *a h b*, D *h* being ordinately applied to the axis *a b*.

Thus a biquadratic equation may be formed, whereby the point *h* ſhall be found, and thence the point D, whoſe diſtance from A is to *h e* as the excentricity of the earth's orbit to half its axis.

Therefore I ſhall only obſerve farther, that here occurs an obvious queſtion, what, in ſo extended a ſearch for principles leading to the ſolution of any problem, as the ancient analyſis admits of, can conduct to the moſt genuine upon each ſeveral occaſion.

But

But for this end, where commodious principles do not readily offer themſelves, the moſt general means is to conſider firſt ſimple caſes of the problem in queſtion, and from thence to proceed gradually to the more complex, as has been here done in the preſent problem, where the ſeveral preceding caſes lead one after another to the points and lines required for the laſt caſe, wherein the problem is ſtated in its moſt extenſive form.

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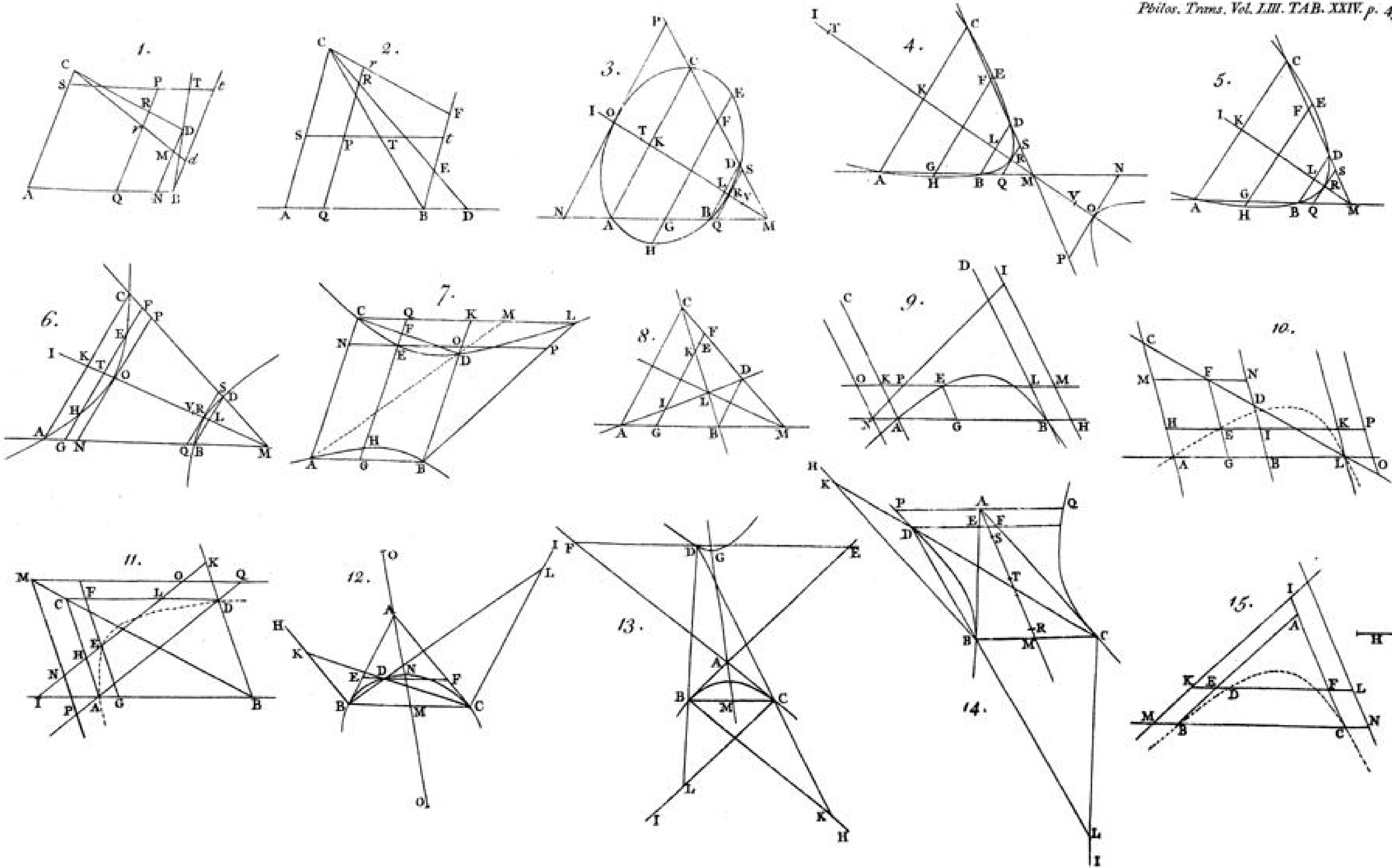
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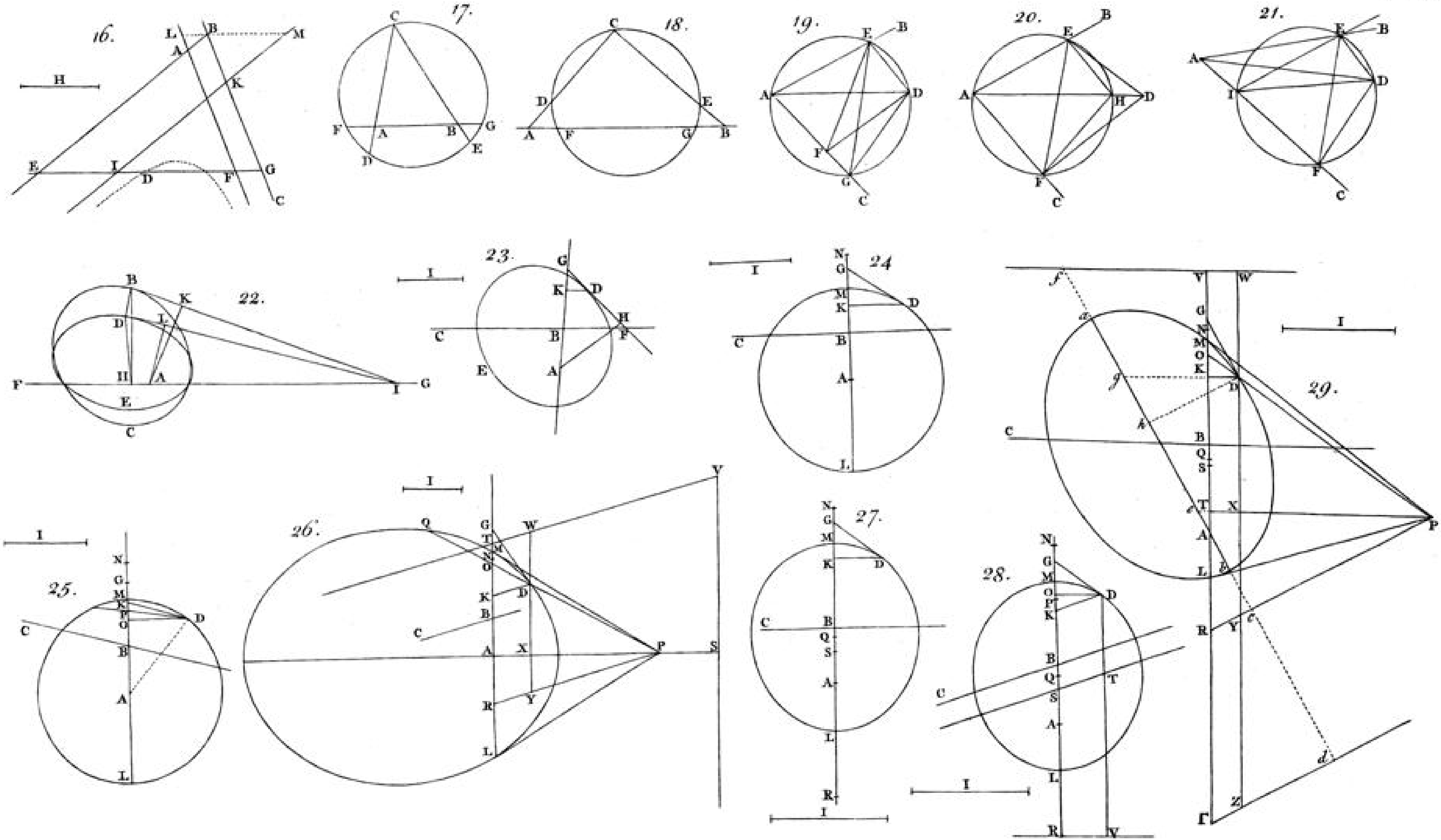
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